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Research Article

Picture Fuzzy Operators with Application in Pattern Recognition

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Abstract. In our daily lives, we face many uncertain situations where vague or imprecise data are the robust issue for consideration. Handling such imprecise data is more difficult than precise data. Picture fuzzy sets are powerful concepts to deal with these kinds of imprecise data effectively because of its diverse operations techniques in several dimensions. In this article, different sorts of operators for picture fuzzy sets are defined and various related properties of these operators are explored. Some mean operators in particular cases are discussed with some of their associated properties. Finally, a real life application is described in decision-making.

Keywords. Picture fuzzy set, Difference operator, Possibility operator, Necessary operator, Closure operator, Interior operator, Mean operator

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1. Introduction

Modern innovations are going through research with data and these tasks become more difficult when dealing with vague data. In 1965, L. A. Zadeh [33] has achieved a great success for introducing the concept of fuzzy set to handle the vague data. Since then, many theories have been developed for treating imprecision and uncertainty. In fuzzy set, every element associated with a membership degree and the complement of the membership degree reflects the non-membership degree of that element. However, much research in application fields has found that the non-membership degree of an element is not always the direct complement of

the membership degree because there arises some hesitation. In 1986, Atanassov [1] developed the concept of intuitionistic fuzzy set which is a generalization of the fuzzy set and the non-membership degree is less than or equal to the complement of the membership degree. Cuong [3], and Cuong and Kreinovich [4] introduced the concept of picture fuzzy set which is a direct extension of fuzzy set and intuitionistic fuzzy set by incorporating the idea of positive, negative and neutral membership degree of an element. Dutta and Ganju [7] introduced (α,δ,β)-cut and strong (α,δ,β)-cut, level set for picture fuzzy set and discussed some properties of them. Some algebraic operations such as algebraic sum, algebraic product, scalar multiplication, exponential operation for picture fuzzy sets are discussed by Silambarasan [22]. He also described arithmetic mean operation for picture fuzzy sets. The averaging operators on picture fuzzy sets along their applications are also becoming a great attention by numerous researchers. Khan *et al*. [14] investigated the information aggregation operators' method under the picture fuzzy environment with the help of Einstein norms operations and applied a group decision making problem. Wei [32] discussed the multiple attribute decision making problem based on the arithmetic, geometric aggregation operators and Hamacher operations of picture fuzzy. Luo and Long [15] studied picture fuzzy geometric aggregation operators based on a trapezoidal fuzzy number and applied it to multi-attribute decision making and pattern recognition. Hasan *et al*. [10] discussed some picture fuzzy mean operators of picture fuzzy sets and applied in decision-making problem. Picture fuzzy aggregation operators are also discussed some researchers, e.g., Garg [9], Jana *et al*. [11], Qiyas *et al*. [20], Tian *et al*. [28], Wang *et al*. [30], and Wei [31]. Besides these some other operations of picture fuzzy sets are also depicted by Chau *et al*. [2], Dinh and Thao [5], Dogra and Pal [6], Ganie *et al*. [8], Kadian and Kumar [12], Khan *et al*. [13], Luo and Zhang [16], Meksavang *et al*. [17], Nguyen *et al*. [18], Peng and Dai [19], Si *et al*. [21], Singh *et al*. [23], Singh and Ganie [24], Son [25], Son *et al*. [26], Thao [27], and Wang [29].

In this article, different types of operators such as difference operator, necessity operator, possibility operator, closure operator, interior operator for picture fuzzy set are defined and various related properties of these operators are explored. Finally, some mean operators in particular cases are discussed with a real life application.

The article is organized as follows: In Section 2, some basic definitions and operations are given essential to the rest of the paper. In Section 3, different kinds of operators for picture fuzzy set are discussed. In Section 4, the mean operators for picture fuzzy sets in some particular cases are discussed. In Section 5, a multi-attribute decision making problem is discussed. In Section 6, the conclusion remark is given.

2. Preliminaries

Definition 2.1 ([33])**.** A FS A in U is given by $A = \{(\theta,\xi_A(\theta)) : \theta \in U\}$, with $\xi_A : U \to [0,1]$ and $0 \le \xi_A(\theta) \le 1,\ \forall\ \alpha \in U$.

Definition 2.2 ([1])**.** An IFS A in U is defined by $A = \{(\theta,\xi_A(\theta),\zeta_A(\theta)) : \theta \in U\}$, with $\xi_A : U \to [0,1]$, $\zeta_A : U \to [0,1]$ and $0 \le \xi_A(\theta) + \zeta(\theta) \le 1,\ \forall\ \theta \in U$ and the hesitant value, $\pi_A(\theta) = 1 - (\xi_A(\theta) + \zeta_A(\theta))$.

Definition 2.3 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *necessity operator of* A is given by $\Box A = \{(\theta, \xi_A(\theta), 1 - \xi_A(\theta)) : \theta \in U\}$.

Definition 2.4 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *possibility operator of* A is given by $\Diamond A = \{(\theta, 1 - \zeta_A(\theta), \zeta_A(\theta)) : \theta \in U\}$.

Definition 2.5 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *closure operator of* A is given by

$$C(A) = \{(\theta, P, Q) : \theta \in U\},$$

where $P = \max\limits_{u \in U} \xi_A(u)$ and $Q = \min\limits_{u \in U} \zeta_A(u)$.

Definition 2.6 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *interior operator of* A is given by

$$I(A) = \{(\theta, p, q) : \theta \in U\},$$

where $p = \min\limits_{u \in U} \xi_A(u)$ and $q = \max\limits_{u \in U} \zeta_A(u)$.

Definition 2.7 ([4, 11]). A PFS A on U is given by $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ with $\xi_A(\theta), \tau_A(\theta), \zeta_A(\theta) \in [0,1]$ and $0 \le \xi_A(\theta) + \tau_A(\theta) + \zeta_A(\theta) \le 1$, $\forall\ \theta \in U$ and the refusal value, $\pi_A(\theta) = 1 - (\xi_A(\theta) + \tau_A(\theta) + \zeta_A(\theta))$, $\forall\ \theta \in U$.

In the whole paper, PFS(U) represents the collection of all PFSs in U.

Definition 2.8 ([3, 4]). Let $X, Y \in \text{PFS}(U)$, then

(i) $X \subseteq Y$ iff $\forall\ \theta \in U$, $\xi_X(\theta) \le \xi_Y(\theta)$, $\tau_X(\theta) \le \tau_Y(\theta)$ and $\zeta_X(\theta) \ge \zeta_Y(\theta)$;

(ii) $X = Y$ iff $\forall\ \theta \in U$, $\xi_X(\theta) = \xi_Y(\theta)$, $\tau_X(\theta) = \tau_Y(\theta)$ and $\zeta_X(\theta) = \zeta_Y(\theta)$;

(iii) $X \cup Y = \{(\theta, \max(\xi_X(\theta), \xi_Y(\theta)), \min(\tau_X(\theta), \tau_Y(\theta)), \min(\zeta_X(\theta), \zeta_Y(\theta))) : \theta \in U\}$;

(iv) $X \cap Y = \{(\theta, \min(\xi_X(\theta), \xi_Y(\theta)), \min(\tau_X(\theta), \tau_Y(\theta)), \max(\zeta_X(\theta), \zeta_Y(\theta))) : \theta \in U\}$;

(v) $X^c = \{(\theta, \zeta_X(\theta), \tau_X(\theta), \xi_X(\theta)) : \theta \in U\}$.

Definition 2.9 ([22]). Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U, then the *arithmetic mean operator of* A *and* B is denoted by $A@B$ and defined as

$$A@B = \left\{\left(\theta, \frac{\xi_A(\theta) + \xi_B(\theta)}{2}, \frac{\tau_A(\theta) + \tau_B(\theta)}{2}, \frac{\zeta_A(\theta) + \zeta_B(\theta)}{2}\right) : \theta \in U\right\}.$$

Definition 2.10 ([10]). Let $X = (\xi_A, \tau_A, \zeta_A)$ be a picture fuzzy value. Then, the score and accuracy functions are respectively given by $S(X) = \xi_A + \tau_A - \zeta_A$ and $H(X) = \xi_A + \tau_A + \zeta_A$, where $S(X) \in [-1, 1]$ and $H(X) \in [0, 1]$.

Definition 2.11 ([10]). Let $X = (\xi_A, \tau_A, \zeta_A)$ and $Y = (\xi_B, \tau_B, \zeta_B)$.

(i) If $S(X) > S(Y)$, then X is greater than Y, denoted by $X \succ Y$.

(ii) If $S(X) = S(Y)$, then

(a) $H(X) = H(Y)$, indicates that $X = Y$.

(b) $H(X) > H(Y)$, indicates that X is greater than Y, denoted by $X \succ Y$.

3. Operators for Picture Fuzzy Sets

Definition 3.1. Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U, then the *difference between A and B* is denoted by $A \ominus B$, and defined as

$$A \ominus B = \{(\theta, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}.$$

Obviously, for $A, B \in \text{PFS}(U)$, $A \ominus B \in \text{PFS}(U)$.

Theorem 3.2. *Let $A, B \in \text{PFS}(U)$, then*

(i) $A \ominus B = A \cap B^c$,

(ii) $A \ominus B = B \ominus A$ *if* $A = B$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A \ominus B = \{(a, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}.$$

But

$$B^c = \{(\theta, \zeta_B(\theta), \tau_B(\theta), \xi_B(\theta)) : \theta \in U\}$$

$$\implies A \cap B^c = \{(\theta, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}$$

$$= A \ominus B.$$

(ii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A \ominus B = \{(\theta, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}.$$

If $A = B$, then $\xi_A(\theta) = \xi_B(\theta)$, $\tau_A(\theta) = \tau_B(\theta)$ and $\zeta_A(\theta) = \zeta_B(\theta)$, $\forall\ \theta \in U$. □

From this, it is certain that $B \ominus A = A \ominus B$ and the result follows.

Theorem 3.3. *Let $A, B \in \text{PFS}(U)$, then*

(i) $A \ominus A^c = A$,

(ii) $A^c \ominus B^c = B \ominus A$,

(iii) $B^c \ominus A^c = A \ominus B$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$A^c = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}.$$

Now,

$$A \ominus A^c = \{(\theta, \min\{\xi_A(\theta), \xi_A(\theta)\}, \min\{\tau_A(\theta), \tau_A(\theta)\}, \max\{\zeta_A(\theta), \zeta_A(\theta)\}) : \theta \in U\}$$

$$= \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\} = A.$$

(ii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A^c = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : a \in U\}$$

and

$$B^c = \{(\theta, \zeta_B(\theta), \tau_B(\theta), \xi_B(\theta)) : \theta \in U\}.$$

Now,

$$\begin{aligned} A^c \ominus B^c &= \{(\theta, \min\{\zeta_A(\theta), \xi_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\xi_A(\theta), \zeta_B(\theta)\}) : \theta \in U\} \\ &= \{(\theta, \min\{\xi_B(\theta), \zeta_A(\theta)\}, \min\{\tau_B(\theta), \tau_A(\theta)\}, \max\{\zeta_B(\theta), \xi_A(\theta)\}) : \theta \in U\} \\ &= B \ominus A. \end{aligned}$$

(iii): Straightforward. □

Definition 3.4. Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U, then the *necessity operator of* A is denoted by $\Box A$ and defined as

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}.$$

Definition 3.5. Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U, then the *possibility operator of* A is denoted by $\Diamond A$ and defined as

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}.$$

Theorem 3.6. *For every PFS A, we have*

(i) $\overline{\Box \overline{A}} = \Diamond A$,
(ii) $\overline{\Diamond \overline{A}} = \Box A$,
(iii) $\Box A \subset A \subset \Diamond A$,
(iv) $\Box \Box A = \Box A$,
(v) $\Box \Diamond A = \Diamond A$,
(vi) $\Diamond \Box A = \Box A$,
(vii) $\Diamond \Diamond A = \Diamond A$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\overline{A} = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}.$$

Now,

$$\Box \overline{A} = \{(\theta, \zeta_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \zeta_A(\theta)) : \theta \in U\}.$$

Therefore,

$$\overline{\Box \overline{A}} = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}.$$

Again,

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}.$$

Thus, $\overline{\Box \overline{A}} = \Diamond A$.

(ii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\overline{A} = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}.$$

Now,

$$\Diamond \overline{A} = \{(\theta, 1 - \tau_A(\theta) - \xi_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}$$

and so

$$\overline{\Box \overline{A}} = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}.$$

Again,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}.$$

Thus, $\overline{\Diamond \overline{A}} = \Box A$.

(iii) Trivial.

(iv) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}$$

and

$$\Box\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\} = \Box A.$$

(v) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$$

and

$$\begin{aligned}\Box\Diamond A &= \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - 1 + \tau_A(\theta) + \zeta_A(\theta)) : \theta \in U\} \\ &= \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\} = \Diamond A.\end{aligned}$$

(vi) Let $A = \{(\theta, \xi\varphi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}$$

and

$$\begin{aligned}\Diamond\Box A &= \{(\theta, 1 - \tau_A(\theta) - 1 + \tau_A(\theta) + \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\} \\ &= \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\} = \Box A.\end{aligned}$$

(vii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$$

and

$$\Diamond\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\} = \Diamond A. \qquad \square$$

Theorem 3.7. *For every two PFSs A and B,*

(i) $\Box(A \cap B) = \Box A \cap \Box B$,

(ii) $\Box(A \cup B) = \Box A \cup \Box B$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A \cap B = \{\theta, \min(\xi_A(\theta), \xi_B(\theta)), \min(\tau_A(\theta), \tau_B(\theta)), \max(\zeta_A(\theta), \zeta_B(\theta)) : \theta \in U\}.$$

Now,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}$$

and

$$\Box B = \{(\theta, \xi_B(\theta), \tau_B(\theta), 1 - \tau_B(\theta) - \xi_B(\theta)) : \theta \in Ut\}.$$

Therefore,

$$\Box A \cap \Box B = \{\theta, \min(\xi_A(\theta), \xi_B(\theta)), \min(\tau_A(\theta), \tau_B(\theta)), \max((1 - \tau_A(\theta) - \xi_A(\theta)), (1 - \tau_B(\theta) - \xi_B(\theta))) : \theta \in U\}.$$

Again,

$$\begin{aligned}\Box(A\cap B)&=\{\theta,\min(\xi_A(\theta),\xi_B(\theta)),\min(\tau_A(\theta),\tau_B(\theta)),1-\min(\tau_A(\theta),\tau_B(\theta))-\min(\xi_A(\theta),\xi_B(\theta)):\theta\in U\}\\&=\{\theta,\min(\xi_A(\theta),\xi_B(\theta)),\min(\tau_A(\theta),\tau_B(\theta)),\max((1-\tau_A(\theta)-\xi_A(\theta)),(1-\tau_B(\theta)-\xi_B(\theta))):\theta\in U\}\\&=\Box A\cap\Box B.\end{aligned}$$

(ii) Trivial. □

Theorem 3.8. *For every two PFSs A and B,*

(i) $\Box(A@B)=\Box A@\Box B$,

(ii) $\Diamond(A@B)=\Diamond A@\Diamond B$.

Proof. (i) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ and $B=\{(\theta,\xi_B(\theta),\tau_B(\theta),\zeta_B(\theta)):\theta\in U\}$ be two PFSs on U, then

$$A@B=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{\zeta_A(\theta)+\zeta_B(\theta)}{2}\right):\theta\in U\right\}.$$

Now,

$$\Box A=\{(\theta,\xi_A(\theta),\tau_A(\theta),1-\tau_A(\theta)-\xi_A(\theta)):\theta\in U\}$$

and

$$\Box B=\{(\theta,\xi_B(\theta),\tau_B(\theta),1-\tau_B(\theta)-\xi_B(\theta)):\theta\in U\}.$$

Therefore,

$$\begin{aligned}\Box A@\Box B&=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{(1-\tau_A(\theta)-\xi_A(\theta))+(1-\tau_B(\theta)-\xi_B(\theta))}{2}\right):\theta\in U\right\}\\&=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{2+(-\tau_A(\theta)-\xi_A(\theta))+(-\tau_B(\theta)-\xi_B(\theta))}{2}\right):\theta\in U\right\}\\&=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{2-(\xi_A(\theta)+\xi_B(\theta))-(\tau_A(\theta)+\tau_B(\theta))}{2}\right):\theta\in U\right\}\\&=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},1-\frac{\xi_A(\theta)+\xi_B(\theta)}{2}-\frac{\tau_A(\theta)+\tau_B(\theta)}{2}\right):\theta\in U\right\}.\end{aligned}$$

Again,

$$(A@B)=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},1-\frac{\xi_A(\theta)+\xi_B(\theta)}{2}-\frac{\tau_A(\theta)+\tau_B(\theta)}{2}\right):\theta\in U\right\}=A@B.$$

(ii) Straightforward. □

Definition 3.9. Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U, then the *closure operator of A* is denoted by $C(A)$ and defined as

$$C(A)=\{(\theta,P,Q,R):\theta\in U\},$$

where

$$P=\max_{u\in U}\xi_A(u),\ Q=\min_{u\in U}\tau_A(u)\ \text{ and }\ R=\min_{u\in U}\zeta_A(u).$$

Definition 3.10. Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U, then the *interior operator of A* is denoted by $I(A)$ and defined as

$$I(A)=\{(\theta,p,q,r):\theta\in U\},$$

where

$$p = \min_{u\in U} \xi_A(u),\ q = \min_{u\in U} \tau_A(u) \text{ and } r = \max_{u\in U} \zeta_A(u).$$

Clearly, $I(A) \subset A$ and $I(A) \subset C(A)$.

Theorem 3.11. *For every PFSs A and B,*

(i) $C(A)$ *is a PFS,*

(ii) $I(A)$ *is a PFS,*

(iii) $I(A) \subset A$,

(iv) $I(A) \subset C(A)$,

(v) $C(C(A)) = C(A)$,

(vi) $I(I(A)) = I(A)$,

(vii) $\overline{I(\overline{A})} = C(A)$.

Proof. Prats (i) and (ii) are trivial.

(iii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$I(A) = \{(\theta, \min_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \max_{u\in U} \zeta_A(u)) : \theta \in U\}.$$

Clearly,

$$\min_{u\in U} \xi_A(u) \le \xi_A(\theta),\ \min_{u\in U} \tau_A(u) \le \tau_A(\theta) \text{ and } \max_{u\in U} \zeta_A(u) \ge \zeta_A(\theta).$$

Therefore, $I(A) \subset A$.

(iv) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$I(A) = \left\{\left(\theta, \min_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \max_{u\in U} \zeta_A(u)\right) : \theta \in U\right\}$$

and

$$C(A) = \left\{\left(\theta, \max_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \min_{u\in U} \zeta_A(u)\right) : \theta \in U\right\}.$$

Clearly,

$$\min_{u\in U} \xi_A(u) \le \max_{u\in U} \xi_A(u),\ \min_{u\in U} \tau_A(u) \le \min_{u\in U} \tau_A(u) \text{ and } \max_{u\in U} \zeta_A(u) \ge \min_{u\in U} \zeta_A(u).$$

Therefore, $I(A) \subset C(A)$.

(v) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$C(A) = \left\{\left(\theta, \max_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \min_{u\in U} \zeta_A(u)\right) : \theta \in U\right\},$$

$$C(C(A)) = \left\{\left(\theta, \max_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \min_{u\in U} \zeta_A(u)\right) : \theta \in U\right\} = C(A).$$

(vi) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$I(A) = \left\{\left(\theta, \min_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \max_{u\in U} \zeta_A(u)\right) : \theta \in U\right\}$$

and

$$I(I(A)) = \left\{\left(\theta, \min_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \max_{u\in U} \zeta_A(u)\right) : \theta \in U\right\} = I(A).$$

(vii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\begin{aligned}
\overline{A} &= \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\},\\
\overline{I(\overline{A})} &= \overline{I\{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}}\\
&= \overline{\left\{\left(\theta, \min_{u\in U} \zeta_A(u), \min_{u\in U} \tau_A(u), \max_{u\in U} \xi_A(u)\right) : \theta \in U\right\}}\\
&= \left\{\left(\theta, \max_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \min_{u\in U} \zeta_A(u)\right) : \theta \in U\right\}\\
&= C(A).
\end{aligned}$$

□

4. Mean Operators for Picture Fuzzy Sets

Definition 4.1 (Picture Fuzzy Geometric Mean (PFGM))**.** Let $A, B \in \mathrm{PFS}(U)$, then the *geometric mean between A and B* is denoted by $A\$B$, and defined as

$$A\$B = \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}, \sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}, \sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}) : \theta \in U\}.$$

Proposition 4.2. *For any two PFSs A and B, A\$B is also PFS.*

Proof. For $A\$B$,

$$\begin{aligned}
0 &\le \sqrt{\xi_A(\theta)\cdot\xi_B(\theta)} + \sqrt{\tau_A(\theta)\cdot\tau_B(\theta)} + \sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}\\
&\le \frac{\xi_A(\theta)+\xi_B(\theta)}{2} + \frac{\tau_A(\theta)+\tau_B(\theta)}{2} + \frac{\zeta_A(\theta)+\zeta_B(\theta)}{2}\\
&\le \frac{\xi_A(\theta)+\tau_A(\theta)+\zeta_A(\theta)}{2} + \frac{\xi_B(\theta)+\tau_B(\theta)+\zeta_B(\theta)}{2}\\
&\le \frac{1}{2} + \frac{1}{2}\\
&\le 1 \quad (\text{as } 0 \le \xi_A(\theta)+\tau_A(\theta)+\zeta_A(\theta) \le 1).
\end{aligned}$$

□

Theorem 4.3. *For every $A, B, C \in \mathrm{PFS}(U)$, the following hold:*

(i) $A\$A = A$,

(ii) $A\$B = B\$A$,

(iii) $A\$(B \cup C) = (A\$B) \cup (A\$C)$,

(iv) $A\$(B \cap C) = (A\$B) \cap (A\$C)$,

(v) $(A\$B)\$C = (A\$C)\$(B\$C)$,

(vi) $(A^c\$B^c)^c = A\$B$,

(vii) $(A\$B)^c = A^c\$B^c$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\begin{aligned}
A\$A &= \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_A(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_A(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_A(\theta)}) : \theta \in U\}\\
&= \{(\theta, \sqrt{(\xi_A(\theta))^2}\sqrt{(\tau_A(\theta))^2}\sqrt{(\zeta_A(\theta))^2}) : \theta \in U\}\\
&= \{(\theta, \xi_A(\theta)\tau_A(\theta)\zeta_A(\theta)) : \theta \in U\} = A.
\end{aligned}$$

(ii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A\$B = \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}) : \theta \in U\},$$
$$= \{(\theta, \sqrt{\xi_B(\theta)\cdot\xi_A(\theta)}\sqrt{\tau_B(\theta)\cdot\tau_A(\theta)}\sqrt{\zeta_B(\theta)\cdot\zeta_A(\theta)}) : \theta \in U\} = B\$A.$$

(iii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$, $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ and $C = \{(\theta, \xi_C(\theta), \xi_C(\theta), \zeta_C(\theta)) : \theta \in U\}$ be three PFSs on U. Then,

$$B \cup C = \{(\theta, \max\{\xi_B(\theta), \xi_C(\theta)\}, \min\{\tau_B(\theta), \tau_C(\theta)\}, \min\{\zeta_B(\theta), \zeta_C(\theta)\}) : \theta \in U\}.$$

Now,

$$\begin{aligned} A\$(B \cup C) &= \{(\theta, \sqrt{\xi_A(\theta)\cdot\max\{\xi_B(\theta), \xi_C(\theta)\}}\sqrt{\tau_A(\theta)\cdot\min\{\tau_B(\theta), \tau_C(\theta)\}} \\ &\quad \cdot\sqrt{\zeta_A(\theta)\cdot\min\{\zeta_B(\theta), \zeta_C(\theta)\}}) : \theta \in U\} \\ &= \{(\theta, \sqrt{\max\{\xi_A(\theta)\cdot\xi_B(\theta)\xi_A(\theta)\cdot\xi_C(\theta)\}}\sqrt{\min\{\tau_A(\theta)\cdot\tau_B(\theta), \tau_A(\theta)\cdot\tau_C(\theta)\}} \\ &\quad \cdot\sqrt{\min\{\zeta_A(\theta)\cdot\zeta_B(\theta)\zeta_A(\theta)\cdot\zeta_C(\theta)\}}) : \theta \in U\} \\ &= \{(\theta, \max\{\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}, \sqrt{\xi_A(\theta)\cdot\xi_C(\theta)}\} \\ &\quad \cdot\min\{\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}, \sqrt{\tau_A(\theta)\cdot\tau_C(\theta)}\}, \\ &\quad \min\{\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}, \sqrt{\zeta_A(\theta)\cdot\zeta_C(\theta)}\}) : \theta \in U\}. \end{aligned}$$

Now,

$$A\$B = \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}) : \theta \in U\},$$
$$A\$C = \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_C(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_C(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_C(\theta)}) : \theta \in U\}$$

and

$$\begin{aligned} (A\$B)\cup(A\$C) &= \{(\theta, \max\{\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}, \sqrt{\xi_A(\theta)\cdot\xi_C(\theta)}\} \\ &\quad \min\{\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}, \sqrt{\tau_A(\theta)\cdot\tau_C(\theta)}\}, \\ &\quad \min\{\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}, \sqrt{\zeta_A(\theta)\cdot\zeta_C(\theta)}\}) : \theta \in U\}. \end{aligned}$$

Thus, $A\$(B \cup C) = (A\$B) \cup (A\$C)$.

(vii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then, we have

$$A\$B = \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}) : \theta \in U\},$$

which implies

$$(A\$B)^c = \{(\theta, \sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}) : \theta \in U\}.$$

Again,

$$A^c = \{(\theta, \zeta_A(\theta)\tau_A(\theta)\xi_A(\theta)) : \theta \in U\} \quad \text{and} \quad B^c = \{(\theta, \zeta_B(\theta)\tau_B(\theta)\xi_B(\theta)) : \theta \in U\}.$$

Therefore,

$$A^c\$B^c = \{(\theta, \sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}) : \theta \in U\},$$

which implies that

$$(A\$B)^c = A^c\$B^c.$$

□

Theorem 4.4. *For any two PFSs A and B, we have*

(i) *If $A \subset B$, then $A\$B \subset B$,*

(ii) $A\$(A \cup B) \subset A \cup B$,

(iii) $A\$(A \cap B) \subset A$

Proof. Trivial. □

Definition 4.5. Let $A_i \in \mathrm{PFS}(U)$, where $i = 1,2,\ldots,n$. Then, the *geometric mean of* A_i is denoted by $\$(A_i)$, and defined as

$$\begin{aligned}\$(A_i) &= \{(\theta, \sqrt[n]{\xi_{A_1}(\theta)\cdot\xi_{A_2}(\theta)\ldots\xi_{A_n}(\theta)}\sqrt[n]{\tau_{A_1}(\theta)\cdot\tau_{A_2}(\theta)\ldots\tau_{A_n}(\theta)} \\ &\quad \cdot \sqrt[n]{\zeta_{A_1}(\theta)\cdot\zeta_{A_2}(\theta)\ldots\zeta_{A_n}(\theta)}) : \theta \in U\} \\ &= \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}\xi_{A_i}(\theta)}\sqrt[n]{\prod_{i=1}^{n}\tau_{A_i}(\theta)}\sqrt[n]{\prod_{i=1}^{n}\zeta_{A_i}(\theta)}\right) : \theta \in U\right\}.\end{aligned}$$

Proposition 4.6. *For any $A_i \in \mathrm{PFS}(U)$, where $i = 1,2,\ldots,n$, $\$(A_i) \in \mathrm{PFS}(U)$.*

Proof. For $\$(A_i)$,

$$\begin{aligned}0 &\le \sqrt[n]{\prod_{i=1}^{n}\xi_{A_i}(\theta)} + \sqrt[n]{\prod_{i=1}^{n}\tau_{A_i}(\theta)} + \sqrt[n]{\prod_{i=1}^{n}\zeta_{A_i}(\theta)} \\ &\le \frac{\sum_{i=1}^{n}\xi_{A_i}(\theta)}{n} + \frac{\sum_{i=1}^{n}\tau_{A_i}(\theta)}{n} + \frac{\sum_{i=1}^{n}\zeta_{A_i}(\theta)}{n} \\ &\le \frac{1}{n}\sum_{i=1}^{n}[\xi_{A_i}(\theta) + \tau_{A_i}(\theta) + \zeta_{A_i}(\theta)] \\ &\le \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \\ &\le \frac{1+1+\ldots+1}{n} \\ &\le \frac{n}{n} = 1 \quad (\text{as, } 0 \le \xi_{A_i}(\theta) + \tau_{A_i}(\theta) + \zeta_{A_i}(\theta) \le 1).\end{aligned}$$

□

Theorem 4.7. *For every $A, B, C \in \mathrm{PFS}(U)$, the followings hold:*

(i) $\$(A_i) = A_i$;

(ii) $(\$(A_i))^c = \$(A_i^c)$;

(iii) $(\$(A_i^c))^c = \$(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i = \{(\theta, \xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)) : \theta \in U\} \in \mathrm{PFS}(U)$. Then

$$\$(A_i) = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))}\sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}\right) : \theta \in U\right\}.$$

Therefore,

$$(\$(A_i))^c = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))}\right) : \theta \in U\right\}.$$

Again,

$$A_i^c = \{(\theta, \zeta_{A_i}(\theta), \tau_{A_i}(\theta), \xi_{A_i}(\theta)) : \theta \in U\},$$

then we have

$$\$(A_i^c) = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))}\right) : \theta \in U\right\},$$

which implies that,

$$(\$(A_i^c))^c = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))}\sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))}\sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}\right) : \theta \in U\right\} = \$(A_i).$$

Therefore, $(\$(A_i^c))^c = \$(A_i)$. □

Definition 4.8 (Picture Fuzzy Harmonic Mean (PFHM))**.** Let $A, B \in \mathrm{PFS}(U)$, then the *harmonic mean between A and B* is denoted by $A©B$, and defined as

$$A©B = \left\{\left(\theta, \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)}\right) : \theta \in U\right\}.$$

Theorem 4.9. *For every $A, B, C \in \mathrm{PFS}(U)$, the followings hold:*

(i) $A©A = A$;

(ii) $A©B = B©A$;

(iii) $(A^c©B^c)^c = A©B$;

(iv) $(A©B)^c = A^c©B^c$;

(v) $(A©B)©C \neq A©(B©C)$.

Proof. Parts (i), (ii) and (iii) trivial.

(iv) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then, we have

$$A©B = \left\{\left(\theta, \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)}\right) : \theta \in U\right\},$$

which implies

$$(A©B)^c = \left\{\left(\theta, \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)}\right) : \theta \in U\right\}.$$

Again,

$$A^c = \{(\theta, \zeta_A(\theta)\tau_A(\theta)\xi_A(\theta)) : \theta \in U\} \text{ and } B^c = \{(\theta, \zeta_B(\theta)\tau_B(\theta)\xi_B(\theta)) : \theta \in U\}.$$

Therefore,

$$A^c©B^c = \left\{\left(\theta, \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)}\right) : \theta \in U\right\}.$$

Therefore, $(A©B)^c = A^c©B^c$. □

Definition 4.10. Let $A_i \in \mathrm{PFS}(U)$, where $i = 1, 2, \ldots, n$. Then, the *harmonic mean of A_i* is denoted by $©(A_i)$, and defined as

$$©(A_i) = \left\{\left(\theta, \frac{n}{\frac{1}{\xi_{A_1}(\theta)}+\frac{1}{\xi_{A_2}(\theta)}+\ldots+\frac{1}{\xi_{A_n}(\theta)}} \frac{n}{\frac{1}{\tau_{A_1}(\theta)}+\frac{1}{\tau_{A_2}(\theta)}+\ldots+\frac{1}{\tau_{A_n}(\theta)}}\right.\right.$$

$$\left.\frac{n}{\frac{1}{\zeta_{A_1}(\theta)}+\frac{1}{\zeta_{A_2}(\theta)}+\ldots+\frac{1}{\zeta_{A_n}(\theta)}}\right):\theta\in U\Bigg\}$$
$$=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\right):\theta\in U\right\}.$$

Theorem 4.11. *For every $A_i\in\mathrm{PFS}(U)$, where $i=1,2,\ldots,n$. Then the following hold:*

(i) $©(A_i)=A_i$;

(ii) $(©(A_i))^c=©(A_i^c)$;

(iii) $(©(A_i^c))^c=©(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i=\{(\theta,\xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)):\theta\in U\}\in\mathrm{PFS}(U)$. Then,

$$©(A_i)=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\right):\theta\in U\right\}.$$

Again,

$$A_i^c=\{(\theta,\zeta_{A_i}(\theta),\tau_{A_i}(\theta),\xi_{A_i}(\theta)):\theta\in U\},$$

then

$$©(A_i^c)=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\right):\theta\in U\right\},$$

which implies

$$(©(A_i^c))^c=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\right):\theta\in U\right\}=©(A_i).$$

Therefore, $(©(A_i^c))^c=©(A_i)$. □

Definition 4.12 (Picture Fuzzy Arithmetic Mean (PFAM))**.** Let $A_i\in\mathrm{PFS}(U)$, where $i=1,2,\ldots,n$. Then, the *arithmetic mean of* A_i is denoted by $@(A_i)$, and defined as

$$@(A_i)=\left\{\left(\theta,\frac{\xi_{A_1}(\theta)+\xi_{A_2}(\theta)+\ldots+\xi_{A_n}(\theta)}{n},\frac{\tau_{A_1}(\theta)+\tau_{A_2}(\theta)+\ldots+\tau_{A_n}(\theta)}{n},\right.\right.$$
$$\left.\left.\frac{\zeta_{A_1}(\theta)+\zeta_{A_2}(\theta)+\ldots+\zeta_{A_n}(\theta)}{n}\right):\theta\in U\right\}$$
$$=\left\{\left(\theta,\frac{\sum_{i=1}^{n}\xi_{A_i}(\theta)}{n},\frac{\sum_{i=1}^{n}\tau_{A_i}(\theta)}{n}\frac{\sum_{i=1}^{n}\zeta_{A_i}(\theta)}{n}\right):\theta\in U\right\}.$$

Proposition 4.13. *For any $A_i\in\mathrm{PFS}(U)$, where $i=1,2,\ldots,n$, $@(A_i)\in\mathrm{PFS}(U)$.*

Proof. For $@(A_i)$,

$$0\le\frac{\sum_{i=1}^{n}\xi_{A_i}(\theta)}{n}+\frac{\sum_{i=1}^{n}\tau_{A_i}(\theta)}{n}+\frac{\sum_{i=1}^{n}\zeta_{A_i}(\theta)}{n}$$
$$\le\frac{1}{n}\sum_{i=1}^{n}[\xi_{A_i}(\theta)+\tau_{A_i}(\theta)+\zeta_{A_i}(\theta)]$$
$$\le\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}$$

$$\leq \frac{1+1+\ldots+1}{n}$$
$$\leq \frac{n}{n} = 1 \quad (\text{as, } 0 \leq \xi_{A_i}(\theta)+\tau_{A_i}(\theta)+\zeta_{A_i}(\theta) \leq 1). \qquad \square$$

Theorem 4.14. *For every $A,B,C \in \mathrm{PFS}(U)$, then the following hold:*

(i) $@(A_i) = A_i$;

(ii) $(@(A_i))^c = @(A_i^c)$;

(iii) $(@(A_i^c))^c = @(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i = \{(\theta, \xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)) : \theta \in U\} \in \mathrm{PFS}(U)$. Then

$$@(A_i) = \left\{\left(\theta, \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}\right) : \theta \in U\right\}.$$

Therefore,

$$(@A_i)^c = \left\{\left(\theta, \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}\right) : \theta \in U\right\}.$$

Again,

$$A_i^c = \{(\theta, \zeta_{A_i}(\theta)\tau_{A_i}(\theta)\xi_{A_i}(\theta)) : \theta \in U\}$$

and

$$@(A_i^c) = \left\{\left(\theta, \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}\right) : \theta \in U\right\},$$

which implies that,

$$(@(A_i^c))^c = \left\{\left(\theta, \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}\right) : \theta \in U\right\} = @(A_i).$$

Therefore, $(@(A_i^c))^c = @(A_i)$. $\square$

Definition 4.15. Let $A,B \in \mathrm{PFS}(U)$, then the *weighted mean between A and B* is denoted by $A \circledcirc B$, and defined as

$$A \circledcirc B = \left\{\left(\theta, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2} \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2} \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\},$$

where $w_1+w_2 \in [0,1]$ and $0 < w_1+w_2 \leq 1$.

Theorem 4.16. *For every $A,B,C \in \mathrm{PFS}(U)$, the following hold:*

(i) $A \circledcirc A = A$;

(ii) $A \circledcirc B = B \circledcirc A$ *if* $A = B$;

(iii) $(A \circledcirc B)^c = A^c \circledcirc B^c$;

(iv) $(A^c \circledcirc B^c)^c = A \circledcirc B$.

Proof. Parts (i), (ii) and (iii) are trivial.

(iv) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : a \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U.

Then,

$$A \circledcirc B = \left\{\left(\theta, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2} \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2} \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\}.$$

Again,

$$A^c = \{(\theta, \zeta_A(\theta)\tau_A(\theta)\xi_A(\theta)) : \theta \in U\}$$

and

$$B^c = \{(\theta, \zeta_B(\theta)\tau_B(\theta)\xi_B(\theta)) : \theta \in U\},$$

then we have

$$A^c \circledcirc B^c = \left\{\left(\theta, \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}, \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2}, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\},$$

which implies

$$(A^c \circledcirc B^c)^c = \left\{\left(\theta, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2}, \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2}, \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\} = A \circledcirc B.$$

Therefore, $(A^c \circledcirc B^c)^c = A \circledcirc B$. □

Definition 4.17. Let $A_i \in \mathrm{PFS}(U)$, where $i = 1,2,\ldots,n$. Then, the *weighted mean of* A_i is denoted by $\circledcirc(A_i)$, and defined as

$$\begin{aligned}\circledcirc(A_i) = \Bigg\{\Bigg(\theta, &\frac{w_1\xi_{A_1}(\theta)+w_2\xi_{A_2}(\theta)+\ldots\ldots+w_n\xi_{A_n}(\theta)}{w_1+w_2+\ldots\ldots+w_n} \frac{w_1\tau_{A_1}(\theta)+w_2\tau_{A_2}(\theta)+\ldots+w_n\tau_{A_n}(\theta)}{w_1+w_2+\ldots+w_n} \\ &\frac{w_1\zeta_{A_1}(\theta)+w_2\zeta_{A_2}(\theta)+\ldots+w_n\zeta_{A_n}(\theta)}{w_1+w_2+\ldots\ldots+w_n}\Bigg) : \theta \in U\Bigg\} \\ = \Bigg\{\Bigg(\theta, &\frac{\sum_{i=1}^n w_i\xi_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}\Bigg) : \theta \in U\Bigg\},\end{aligned}$$

where $w_i \in [0,1]$ and $0 < \sum_{i=1}^n w_i \le 1$.

Theorem 4.18. *For every* $A_i \in \mathrm{PFS}(U)$, *where* $i = 1,2,\ldots,n$, *the following hold:*

(i) $\circledcirc(A_i) = A_i$;

(ii) $(\circledcirc(A_i))^c = \circledcirc(A_i^c)$;

(iii) $(\circledcirc(A_i^c))^c = \circledcirc(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i = \{(\theta, \xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)) : \theta \in U\} \in \mathrm{PFS}(U)$. Then

$$\circledcirc(A_i) = \left\{\left(\theta, \frac{\sum_{i=1}^n w_i\xi_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}\right) : \theta \in U\right\}.$$

Again,

$$A_i^c = \{(\theta, \zeta_{A_i}(\theta), \tau_{A_i}(\theta), \xi_{A_i}(\theta)) : \theta \in U\},$$

then we have

$$©(A_i^c) = \left\{\left(\theta, \frac{\sum_{i=1}^n w_i\zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}, \frac{\sum_{i=1}^n w_i\tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\xi_{A_i}(\theta)}{\sum_{i=1}^n w_i}\right) : \theta \in U\right\},$$

which implies

$$(\circledcirc(A_i^c))^c = \left\{\left(\theta, \frac{\sum_{i=1}^n w_i\xi_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}\right) : \theta \in U\right\} = \circledcirc(A_i).$$

Therefore, $(\circledcirc(A_i^c))^c = \circledcirc(A_i)$. □

5. Application of the Picture Fuzzy Mean Operators to Pattern Recognition

Multi-attribute decision-making (MADM) problem which aims to determine the best alternative by considering more than one attributes in a selection process. The decision will be performed as per the following steps:

Step 1. Picture Fuzzy Decision Matrix will be formed by the experts considering all the alternatives associated with all the attributes.

Step 2. Find the preference values d_i $(i = 1,2,\ldots,5)$ for the operators PFAM, PFGM and PFHM.

Step 3. Find the scores $S(d_i)$ $(i = 1,2,\ldots,n)$ (using score function) of the overall picture fuzzy values d_i $(i = 1,2,\ldots,n)$.

Step 4. Rank all the alternatives A_i $(i = 1,2,\ldots,n)$ in accordance with the values of $S(d_i)$ $(i = 1,2,\ldots,n)$ and select the best one(s).

If any two scores of $S(d_i)$ and $S(d_j)$ are equal, then calculate the accuracy degrees $H(d_i)$ and $H(d_j)$ (using the accuracy function) of the overall picture fuzzy values d_i and d_j, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(d_i)$ and $H(d_j)$.

Case Study

Suppose a company wants to recruit a potential salesman in the company's sales department. There are five candidates $C = \{C_1, C_2, C_3, C_4, C_5\}$ for this position and the company considers four attributes $A = \{A_1, A_2, A_3, A_4\}$ from these candidates. These attributes are academic results, communication skills, experience and innovative ideas. According to the decision of the recruitment committee (experts), the following picture fuzzy matrix is formed under the four attributes.

Step 1: Establishing the picture fuzzy decision matrix by the recruitment committee (experts) considering all the five candidates associated with all the four attributes in Table 1.

Table 1. Picture Fuzzy Decision Matrix

	A_1	A_2	A_3	A_4
C_1	(0.7, 0.2, 0.1)	(0.6, 0.2, 0.2)	(0.5, 0.2, 0.3)	(0.4, 0.2, 0.3)
C_2	(0.5, 0.1, 0.3)	(0.6, 0.2, 0.2)	(0.7, 0.1, 0.2)	(0.4, 0.3, 0.3)
C_3	(0.5, 0.2, 0.2)	(0.4, 0.2, 0.4)	(0.6, 0.2, 0.2)	(0.3, 0.3, 0.2)
C_4	(0.4, 0.4, 0.2)	(0.5, 0.3, 0.2)	(0.8, 0.1, 0.1)	(0.6, 0.2, 0.2)
C_5	(0.3, 0.2, 0.4)	(0.4, 0.4, 0.2)	(0.5, 0.3, 0.2)	(0.7, 0.2, 0.1)

Step 2: Finding the preference values d_i $(i = 1,2,\ldots,5)$ by the operators PFAM, PFGM and PFHM, we have overall preference values d_i as shown in Table 2.

Table 2. Preference values d_i $(i = 1,2,\ldots,5)$ for the operators PFAM, PFGM and PFHM

	d_1	d_2	d_3	d_4	d_5
PFAM	(0.55, 0.20, 0.23)	(0.55, 0.18, 0.25)	(0.45, 0.23, 0.25)	(0.58, 0.25, 0.18)	(0.48, 0.28, 0.23)
PFGM	(0.54, 0.20, 0.21)	(0.54, 0.17, 0.24)	(0.44, 0.22, 0.24)	(0.56, 0.22, 0.17)	(0.45, 0.26, 0.20)
PFHM	(0.53, 0.20, 0.18)	(0.53, 0.14, 0.24)	(0.42, 0.22, 0.23)	(0.54, 0.19, 0.16)	(0.43, 0.25, 0.18)

Step 3: Finding the scores $S(d_i)$ $(i = 1,2,\ldots,n)$ (using score function) of the overall picture fuzzy values d_i $(i = 1,2,\ldots,n)$ in Table 3.

Table 3. The scores $S(d_i)$ $(i = 1,2,\ldots,5)$ of the overall picture fuzzy values d_i $(i = 1,2,\ldots,5)$

	$S(d_1)$	$S(d_2)$	$S(d_3)$	$S(d_4)$	$S(d_5)$
PFAM	0.52	0.58	0.43	0.65	0.53
PFGM	0.53	0.47	0.42	0.61	0.51
PFHM	0.55	0.43	0.41	0.57	0.50

Step 4: Ranking all the candidates C_i $(i = 1,2,3,4,5)$ in accordance with the values of $S(d_i)$ $(i = 1,2,3,4,5)$ and selecting the best one(s) in Table 4.

Table 4. Ranking all the candidates C_i $(i = 1,2,3,4,5)$ in accordance with the values of $S(d_i)$ $(i = 1,2,3,4,5)$ found in Table 3

Operators	Ranking	Best Candidates
PFAM	$C_4 \succ C_2 \succ C_5 \succ C_1 \succ C_3$	C_4
PFGM	$C_4 \succ C_1 \succ C_5 \succ C_2 \succ C_3$	C_4
PFHM	$C_4 \succ C_1 \succ C_5 \succ C_2 \succ C_3$	C_4

Hence, the candidate C_4 is the best and selected for the potential salesman in that company.

6. Conclusions

Picture fuzzy set is the generalization of intuitionistic fuzzy set and fuzzy set and capable to successfully handle the uncertain data. In this article, several types of operators such as difference operator, necessity operator, possibility operator, closure operator, interior operator and some mean operators for picture fuzzy sets are defined. Also, some related properties of these operators are established. Finally, a decision making problem is discussed to demonstration how these operators work in real life situation.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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